Informal Intuition: Necessity of infinite-dimensionality of the range in Theorem 2.4

This is an informal discussion with figures of why the range in $C(X, Y)$ must be infinite-dimensional for the genericity of the subset $\mathcal{E}^{**}(X, Y)$ of "strong" embeddings to obtain. This result contrasts to the case when the subset in question is just a subset of "plain" embeddings, i.e. continuous injective functions. The latter subset is generic whenever Y is $(2n + 1)$-dimensional where n is the dimension of X.[1]

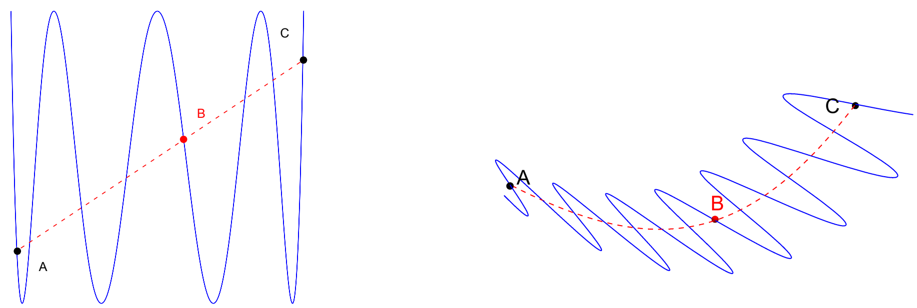
We illustrate the distinction between injectivity and (**) by means of some diagrams. The functions in these diagrams should be interpreted as abstract examples for the two properties, without reference to any belief functions. Each diagram shows the image of some function that has domain $X = [0, 1]$ and either a two- or a three-dimensional range. Because the unit interval has dimension one, the embedding theorem asserts that any continuous function from the unit interval to $\mathbb{R}^3$ can be approximated by a sequence of continuous *injective* functions from the unit interval to $\mathbb{R}^3$. Thus, the function which generates the solid blue loop in the plane in Figure 1 is *not* injective because the point of self-intersection corresponds to the value of the function under two distinct points in its domain. However, as indicated by the red "detour", the third dimension provides ample room for perturbations of this function that are injective.

(a) Image of $[0, 1]$ in $\mathbb{R}^2$ (b) Image of $[0, 1]$ in $\mathbb{R}^3$

Figure 1: Denseness of injectivity

In Figure 2, the solid blue line in the left half of the figure is the image of a sinusoid function taking values in a plane. The sinusoid is injective, but fails to satisfy condition

[1]For a detailed discussion and the proof of the classical embedding theorem (involving "plain" embeddings) see either Hurewicz and Wallman (1941) or Engelking (1989).



(a) Image of $[0,1]$ in $\mathbb{R}^2$

(b) Image of $[0,1]$ in $\mathbb{R}^3$

Figure 2: Denseness of (**) ?

(**) because its value at point B is a convex combination of points A and C on the red dashed line (and of course any other point on the crossing of the red line and blue line is a convex combination of ”extreme” points A and C). The right half of the figure shows that, in this case, too, an additional dimension can provide room for perturbations of the sinusoid that satisfy condition (**). The ”curved” sinusoid in part (b) of Figure 2 can be seen as an element of a sequence of functions approximating the ”flat” sinusoid” in part (a). Each element of the sequence of ”curved” sinusoids satisfies condition (**) even though the ”flat” sinusoid does not.

However, there is a difference. In Figure 1, approximation of the non-injective function by injective functions is obtained by *local* perturbations. In Figure 2, the approximation by functions satisfying condition (**) is based on *global* perturbations. The version of condition (**) in this abstract setting hinges on comparing the value of $\varphi_f(\delta_{\bar{x}})$ with the value of $\varphi_f(\mu)$ for any μ, a comparison which depends on $\hat{x}$ with $\varphi_f(\delta_{\hat{x}})$ far away from $\varphi_f(\delta_{\bar{x}})$.[2]

For injectivity, the local nature of the required perturbations explains why embeddings are dense if the dimension of the range is greater than twice the dimension of the domain. If the domain is the unit interval, as was assumed for Figure 1, then at any point of self-intersection, the range behaves locally like a plane, which leaves another dimension to provide room for perturbations that are injections.

For condition (**), this kind of reasoning is unavailable. Figure 3 gives an example

[2] Using the objects of the paper: condition (**) hinges on comparing the value of $\varphi_{b_i}(\delta_{\bar{t}_i})$ with the value of $\varphi_{b_i}(\mu)$ for any μ, a comparison which depends on types $\hat{t}_i$ with $\varphi_{b_i}(\delta_{\hat{t}_i})$ far away from $\varphi_{b_i}(\delta_{\bar{t}_i})$.

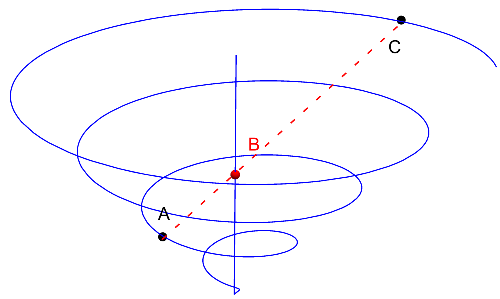


Figure 3: Image of $[0,1]$ in $\mathbb{R}^3$: Failure of denseness of (**)

with a mapping of $[0,1]$ into $\mathbb{R}^3$ whose image is the union of a vertical line and an expanding spiral around the vertical line so that the vertical line belongs to the convex hull of the expanding spiral, which is a non-negligible subset of $\mathbb{R}^3$. In this case there is no sequence of functions from $[0,1]$ to $\mathbb{R}^3$ that satisfy condition (**) and approximate the given function – for any such function, the part that approximates the vertical line inside the spiral must belong to the convex hull of the part that approximates the spiral.

In Supplementary Material to Gizatulina and Hellwig (2017), we show more generally that, for any n, there is a function from $[0,1]$ to $\mathbb{R}^n$ that violates condition (**) and that cannot be approximated by functions from $[0,1]$ to $\mathbb{R}^n$ that satisfy condition (**). If the convex hull of the range of a function from $[0,1]$ to $\mathbb{R}^n$ is a non-negligible subset of $\mathbb{R}^n$, one needs yet another dimension (at least) for the perturbations that satisfy condition (**) and approximate the given function. This is why the proof of Theorem 2.4 makes essential use of the fact that $M^\lambda(T_{-i})$ is infinite-dimensional. By contrast, in the argument of Gizatulina and Hellwig (2014), this fact matters only because, with an infinite-dimensional range, the dimensionality condition of the embedding theorem is satisfied no matter what the (finite) dimension of the domain may be.

References

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