Supplementary Material for:
"The Generic Possibility of Full Surplus Extraction in Models with Large Type Spaces"

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Necessity of infinite-dimensionality of the range in Theorem 2.4

For the convenience of the reader, we begin with a restatement of Theorem 2.4 in Gizatulina and Hellwig (2017) in more abstract terms.

THEOREM 1 *Let X and Y be nonempty compact metric spaces. Assume that the topology on $\mathcal{M}(Y)$ is induced by a metric that is a convex function on $\mathcal{M}(Y) \times \mathcal{M}(Y)$. Let $\mathcal{E}^*(X_i, \mathcal{M}(Y))$ be the set of continuous functions b from X to $\mathcal{M}(Y)$ that satisfy condition (*), i.e., the requirement that, for any $x \in X$ and any $\mu \in \mathcal{M}(X)$,*

$$\varphi_b(\mu) = \varphi_b(\delta_x) \text{ implies } \mu = \delta_x,$$

where δ_x is the degenerate measure that assigns all mass to the singleton $\{x\}$ and $\varphi_b : \mathcal{M}(X) \to \mathcal{M}(Y)$ is defined so that, for any $\mu \in \mathcal{M}(X)$

$$\varphi_b(B|\mu) = \int b(B|x)\mu(dx)$$

for all measurable $B \subset Y$. If Y has more than finitely many elements, then $\mathcal{E}^*(X_i, \mathcal{M}(Y))$ is a residual subset of the space $\mathcal{C}(X, \mathcal{M}(Y))$, i.e., $\mathcal{E}^*(X, \mathcal{M}(Y))$ contains a countable intersection of open and dense subsets of $\mathcal{C}(X, \mathcal{M}(Y))$.

The following proposition shows that the genericity claim made in this theorem is false if Y is a finite set and $\mathcal{M}(Y)$ is a finite-dimensional. This contrasts with the genericity of embeddings, which by the classical embedding theorem holds whenever the dimension of the range is more than twice the dimension of the domain of the functions under consideration. The following result provides a formal statement.

PROPOSITION 2 *Let X be a metric space with more then N elements. Let $C(X,[0,1]^N)$ be the space of continuous functions from X to $[0,1]^N$, endowed with the uniform topology. There exists an open subset $\mathcal{U}$ of $\mathcal{C}(X,[0,1]^N)$ such that, for every $f \in \mathcal{U}$, there exists $x \in X$ such that*

$$f(x) \in co\{f(X \backslash \{x\})\}, \tag{1}$$

i.e., the value of the function at x belongs to the convex hull of its values at points other than x.

COROLLARY 3 *For X as specified in Proposition 2, let $\mathcal{E}^*(X,[0,1]^N)$ be the subset of those functions $f \in \mathcal{C}(X,[0,1]^N)$ for which there exists no $x \in X$ for which (1) holds. Then the set $\mathcal{E}^*(X,[0,1]^N)$ is not dense in $\mathcal{C}(X,[0,1]^N)$.*

Proof of Proposition 2. Choose $p_1,...,p_{N+2}$ in $[0,1]^N$ so that $p_1,...,p_{N+1}$ are in general position[1] and

$$p_{N+2} = \sum_{i=1}^{N+1} \alpha_i p_i, \tag{2}$$

where $\sum_{i=1}^{N+1} \alpha_i = 1$ and $\alpha_i > 0$ for all i.

Fix a function $g \in C(X,[0,1]^N)$ such that, for $i = 1,...,N+2$,

$$g(x_i) = p_i. \tag{3}$$

Then, obviously,

$$g(x_{N+2}) \in co\{g(x_1),...,g(x_{N+1})\}; \tag{4}$$

In fact, since $\alpha_i > 0$ for all i, $g(x_{N+2}) = p_{N+2}$ belongs to the interior of $co\{g(x_1),...,g(x_{N+1})\}$, i.e., there exists $\varepsilon > 0$ such that the ε-ball $B^\varepsilon(p_{N+2})$ around $g(x_{N+2}) = p_{N+2}$ is fully contained in the convex hull of $g(x_1) = p_1,...,g(x_{N+1}) = p_{N+1}$.

Now, fix $\eta = \frac{\varepsilon}{2}$ and let $\mathcal{U}$ be the open η-neighbourhood of g, i.e. the set of all functions f such that $d(f(x),g(x)) < \eta$ for all $x \in X$, where d is the metric on $[0,1]^N$. We claim that, for any $f \in \mathcal{U}$,

$$f(x_{N+2}) \in co\{f(x_1),...,f(x_{N+1})\}. \tag{5}$$

[1] r distinct vertices are in general position in $[0,1]^N$, if there are no $m+2$ vertices ($m = 1,...,N-1$) among them that lie in an m-dimensional linear subspace of $[0,1]^N$.

To prove this claim, we will show that the $\frac{\varepsilon}{2}$-ball $B^{\frac{\varepsilon}{2}}(p_{N+2})$ around $g(x_{N+2}) = p_{N+2}$ is a subset of the convex hull of $f(x_1), ..., f(x_{N+1})$, i.e., that

$$B^{\frac{\varepsilon}{2}}(p_{N+2}) \subset co\{f(x_1), ..., f(x_{N+1})\}. \tag{6}$$

(6) implies (5) because, for $f \in \mathcal{U}$, $d(f(x_{N+2}), g(x_{N+2})) < \eta = \frac{\varepsilon}{2}$, and therefore, $f(x_{N+2}) \in B^{\frac{\varepsilon}{2}}(p_{N+2})$.

To prove (6), we first note that

$$co\{p_1, ..., p_{N+1}\} \subset B^{\eta}(co\{f(x_1), ..., f(x_{N+1})\}). \tag{7}$$

To see this, observe that, for any $q \in co\{p_1, ..., p_{N+1}\}$, there exist $\alpha_i^q, i = 1, ..., N+1$, such that

$$q = \sum_{i=1}^{N+1} \alpha_i^q p_i.$$

For $f \in \mathcal{U}$, the distance between q and the element $\sum_{i=1}^{N+1} \alpha_i^q f(x_i)$ of $co\{f(x_1), ..., f(x_{N+1})\}$ is less than η, i.e. $q \in B^{\eta}(\{co\{f(x_1), ..., f(x_{N+1})\}\})$.

Since $B^{\varepsilon}(p_{N+2}) \subset co\{p_1, ..., p_{N+1}\}$, it follows that

$$B^{\varepsilon}(p_{N+2}) \subset B^{\eta}(co\{f(x_1), ..., f(x_{N+1})\}). \tag{8}$$

Now (6) follows because $\eta = \frac{\varepsilon}{2}$. For suppose that (6) fails so that there exists some $v \in B^{\frac{\varepsilon}{2}}(p_{N+2}) \backslash co\{f(x_1), ..., f(x_{N+1})\}$. Let y be the element of $co\{f(x_1), ..., f(x_{N+1})\}$ that is closest to v, and let

$$z = v + \delta(v - y) \tag{9}$$

where δ is chosen so that $d(z, v)$, the distance between z and v, is exactly η. By the triangle inequality,

$$d(p_{N+2}, z) \leq d(p_{N+2}, v) + d(v, z).$$

By construction, $d(v, z) = \eta = \frac{\varepsilon}{2}$ and $d(p_{N+2}, v) \leq \frac{\varepsilon}{2}$. Thus, $z \in B^{\varepsilon}(p_{N+2})$. By (8), it follows that there exists $u \in co\{f(x_1), ..., fx_{N+1})\}$ such that

$$d(z, u) \leq \eta. \tag{10}$$

Using (9), we obtain

$$d(z, u) = d(v + \delta(v - y), u) = (1 + \delta)d(v, \lambda u + (1 - \lambda)y),$$

where $\lambda := \frac{1}{1+\delta}$. By the definition of y and another application of (9), it follows that

$$d(z,u) \geq (1+\delta)d(v,y) > \delta d(v,y) = d(z,v) \tag{11}$$

Upon combining (10) and (11), we find that $d(z,v) < \eta$, contrary to the assumption that $d(z,v) = \eta$. The assumption that (6) fails has thus led to a contradiction and must be false. ■